MTH 406: Differential geometry of curves and surfaces

Homework VII: Second fundamental form and Minimal surfaces

Problems for practice

1. Establish all the assertions in 3.3 (ii) - (iv), 3.4 (ix).

2. Compute the first and second fundamental forms, the mean and the Gaussian curvature for the following surfaces.

   (a) The *monkey saddle* given by
   $$S = \{(u, v, u^3 - 3v^2u) : u, v \in \mathbb{R}^2\}.$$

   (b) The Möbius strip M in 2.4 (vii) of the Lesson Plan.

   (c) The *Enneper's surface* given by
   $$S = \{(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2) : u, v \in \mathbb{R}^2\}.$$

   (d) The *pseudosphere* obtained by revolving the *tractrix* given by
   $$\gamma(t) = (\sin(t), 0, \cos(t) + \ln(\tan(t/2))),\ t \in (\pi/2, \pi),$$
   about the z-axis.

   (e) The catenoid and the helicoid given in 3.4 (ix) of the Lesson Plan.

3. Classify the surfaces of revolution with constant Gaussian curvature zero and -1.

4. Prove that the mean curvature at a point p of an oriented regular surface S equals the average value of the normal curvature at p over the circle of unit-length directions in $T_p(S)$.

5. Show that every compact regular surface has a point of positive Gaussian curvature.

6. Show that any minimal surface satisfies $K \leq 0$. Conclude that a minimal surface is not compact.

7. Show that if a connected minimal surface has constant Gaussian curvature zero, then it the subset of a plane.

8. Show that the following surfaces are minimal.

   (a) The *Scherk's surface* $f(x, y) = \frac{\cos(y)}{\cos(x)}$ whose domain is chosen appropriately.

   (b) The *Catalan's surface* given by the parametrization
   $$f(u, v) = (u - \sin(u)\cosh(v), 1 - \cos(u)\cosh(v), -4\sin(u/2)\sinh(v/2)).$$